Physics of Fluids (WBPH042-05C)
April 6, 2021, 8:30-11:30

This is an open-book exam, so you are allowed to consult the book, the tutorial exercises, your notes, etc. There are 4 questions in total.

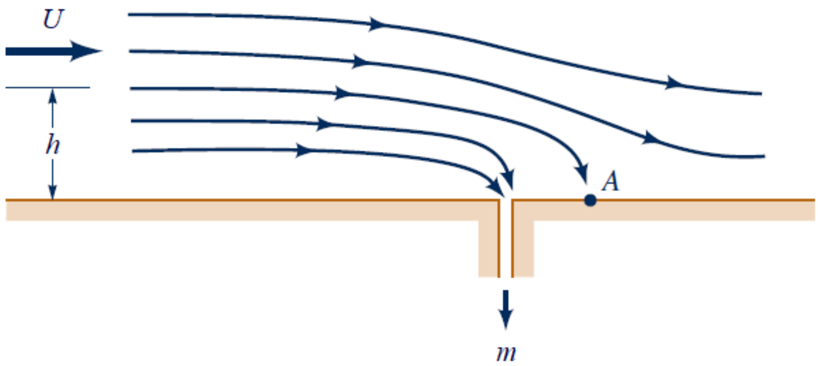
When for some reason you are unable to answer part of a question (a, b, etc.), make a realistic assumption and use this for the rest of the question.

Please write your name, student number and course code (found at the top of this page) *on each answer sheet* that you hand in.

GOOD LUCK!

Exercise 1. Drained ideal plate flow [20pt]

Consider an ideal water flow over a flat plate with a uniform velocity U (m/s), as shown in the figure below. A pump is placed underneath the plate to drain out water through a narrow slit at a volumetric flow rate m (m^2/s) per unit width of the opening.



On the assumption that the fluid is incompressible and inviscid, the flow can be represented by the combination of a uniform flow and a sink. The velocity potential associated with this flow, in Cartesian coordinates, is:

$$\varphi(x, y) = Ux - \frac{m}{2\pi} \ln \sqrt{x^2 + y^2}.$$

Note: Let U be the last digit of your student number, and set m equal to 0.1 times the second last digit of your student number (both in the appropriate units, of course). If either of the digits is zero, use 10 as the value.

a) [3pt] Find the velocity components in the radial and tangent direction (u_r and u_θ, respectively).

b) [4pt] Derive the stream function ψ for this flow. Show that it can be written as
$$\psi(r, \theta) = Ur \sin\theta - \frac{m}{2\pi}\theta + \psi_0,$$
with ψ_0 being constant.

c) [3pt] Find the position of the stagnation point on the plate (point A).

d) [5pt] Determine the critical height h above the plate (far upstream), such that the fluid does not get sucked into the slit.

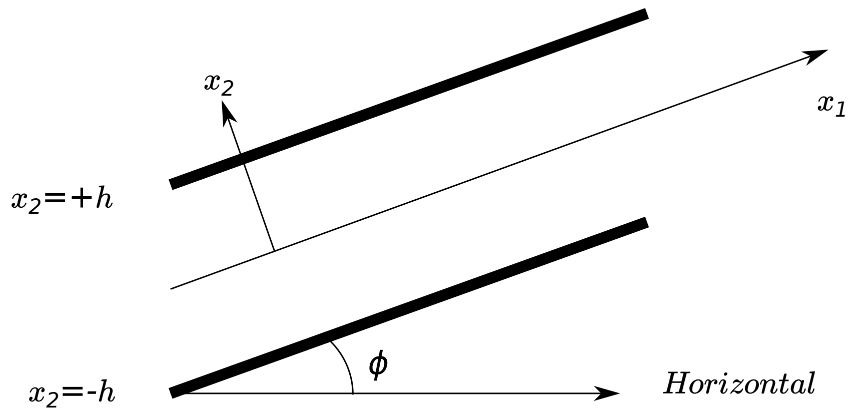
e) [5pt] Assume the pressure far upstream is p_∞. Determine the pressure at the point P located at $(r_P, \theta_P) = (a, b)$. You can ignore the effects of gravity and assume the density of the fluid ρ to be constant.

Exercise 2. Upward pumping cavity flow [20pt]

An incompressible fluid with density ρ is being pumped into a cavity between two parallel walls, located at $x_2 = \pm h$, respectively, as shown below. The walls are inclined at an angle ϕ to the horizontal.

The flow is being driven between the walls in an upward direction by a constant nonzero pressure gradient, $\partial p/\partial x_1 = \text{const.}$, where the x_1 direction lies parallel to the walls, in the direction of flow.

An additive to the liquid causes the viscosity μ to vary in the x_2 direction.



a) [2pt] What is the minimum pressure gradient, $\partial p/\partial x_1$, for which the fluid will flow uphill?

b) [2pt] State suitable boundary conditions for the system.

c) [8 pt] By assuming steady, fully-developed flow, the solution can be written as $\mathbf{u} = (u_1(x_2), 0, 0))$. Determine $u_1(x_2)$ when $\mu = \mu_0(1 + \gamma(x_2/h)^2)$ by solving the Navier-Stokes equations.

d) [3 pt] What is the shear stress felt on the upper wall?

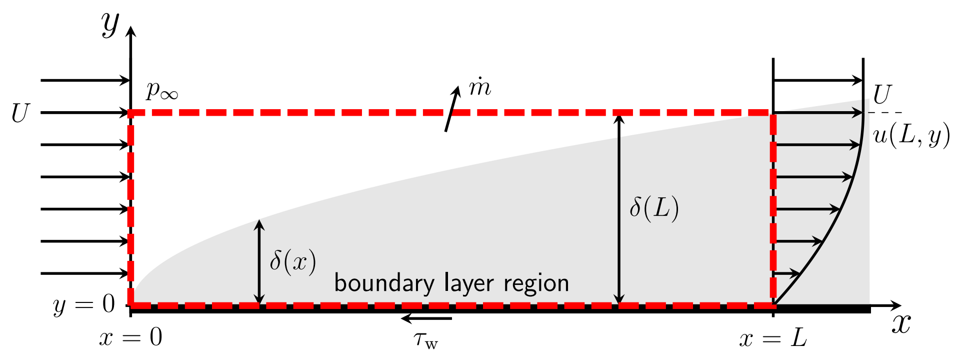
e) [5 pt] What is the volume flow rate per unit depth between the walls when $\gamma = 0$?
Hint: You need to re-evaluate the Navier-Stokes equations with the new function for μ.

Exercise 3. Skin friction on a flat plate [15pt]

Consider a two-dimensional steady flow of a fluid with density ρ and viscosity μ, moving with speed U over a horizontal plate. The horizontal velocity component of the flow field is schematically depicted in the figure below. The skin friction between the fluid and the plate leads to a wall shear stress τ_w and a region of low velocity, a so-called boundary layer. The thickness of the boundary layer region is $\delta(x)$, and the horizontal component of the velocity above the plate is

$$u(x,y) = \begin{cases} U\left[1-\left(\dfrac{\delta(x)-y}{\delta(x)}\right)^2\right] & \text{if } 0 \le y < \delta(x), \\ U & \text{otherwise.} \end{cases}$$

We are going to analyze the problem using the control volume that is highlighted by the red dashed line in the figure below. Its horizontal length is L, and its height is $\delta_L := \delta(L)$. In this exercise, assume the pressure p_∞ to be uniform and ignore gravity.



a) [3pt] Use Buckingham-Π theory to find the dimensionless groups for the parameters U, ρ, μ, L, δ_L, τ_w and $\dot{m}$, where $\dot{m} = \int_\mathrm{top} \rho \mathbf{u}\cdot\mathbf{n}\,dA$ is the mass flux through the top boundary of the control volume.

   Note: In two dimensions, you must express $\dot{m}$ in units of mass flux *per unit depth into the page.*

b) [4pt] Use conservation of mass to find the mass flux $\dot{m}$ in terms of the other parameters. Put your result in terms of the dimensionless groups of part (a).

c) [4pt] Use conservation of x-momentum to find the wall shear stress τ_w in terms of the other parameters. Put your result in terms of the dimensionless groups of part (a).

d) [4pt] The wall shear stress τ_w found in (c) must, by definition, be equal to $\mu\partial u/\partial y$ evaluated at $(x,y) = (L,0)$. Use this to find an expression for δ_L. Put your result in terms of the dimensionless groups of part (a).

Exercise 4. Kinematics [15pt]

In a two-dimensional flow, the Cartesian trajectory of a particle released at the point (x_0, y_0) at time $t = 0$ is given by

$$x(t) = x_0 A^{-t/\tau} \cos\left(\frac{2\pi t}{T}\right),$$
$$y(t) = y_0 + y_0 A^{-t/\tau} \sin\left(\frac{2\pi t}{T}\right),$$

where $A > 0$ is a constant, and $\tau > 0$, $T > 0$ are time scales.

a) [4pt] Determine the Lagrangian particle velocity components $u(t) = dx/dt$ and $v(t) = dy/dt$ and convert these into Eulerian velocity components $u(x, y)$ and $v(x, y)$. In particular, show that the Eulerian velocity components are

$$u(x, y) = -ax - b(y - y_0),$$
$$v(x, y) = -a(y - y_0) + cx,$$

and write a, b and c in terms of A, τ and T.

Note: If you did not manage to solve (a) completely, you may assume that $b > 0$ and $c > 0$ in the following.

b) [4pt] Show that the particle acceleration $a_x = d^2x/dt^2$ is equal to the material derivative D/Dt of the Eulerian velocity component $u(x, y)$.

c) [4pt] Compute the strain rate and rotation tensors for the velocity field you found in (a). Explain in your own words how you can relate the different entries in each of the tensors to the motion of a fluid element inside the flow.

d) [3pt] Answer the following questions by providing clear explanations. Where possible, support your arguments with calculations. Discuss whether the answer depends on the value of A (or a, if you did not solve part (a)).

  i) Is the flow steady?
  ii) Is the flow irrotational?
  iii) Is the flow incompressible?

Useful equations

Derivative of a^x	$\frac{d}{dx}a^x = \ln(a)\, a^x$
Integral of $\frac{1}{a^2+x^2}$	$\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$